ELSEVIER
Linear Algebra and its Applications 346 (2002) 97–107
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Pole-shifting for linear systems over commutative rings ☆

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Received 28 March 2000; accepted 15 September 2001

Submitted by R.A. Brualdi

Abstract

This paper deals with the pole-shifting problem for non-necessarily reachable linear systems. The notion of PS ring is introduced in the same way as the notion of pole assignable ring is given for reachable systems. We prove that a bcs ring is a PS ring and that over a Prüfer domain these properties are equivalent. Finally, we study when the converse of the pole-shifting theorem is verified. © 2002 Elsevier Science Inc. All rights reserved.

AMS classification: 93B55; 93B25; 15A18

Keywords: Linear systems theory; Pole assignability; Bcs ring

1. Introduction and notations

This paper deals with the problem of modifying by feedback the characteristic polynomial of a linear dynamical system over a commutative ring R. In [2,5,10], it is proved that a bcs ring (i.e., every basic submodule of a finitely generated projective R-module M contains a rank one summand of M) is a PA ring (i.e., every reachable linear system is pole assignable). The obstruction for a PA ring to be a bcs ring is

☆ Partially supported by DGICYT PB-0603-C02 and Junta de Castilla y León LE-36/98.
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PII: S0024-3795(01)00502-X

that a basic submodule of a finitely generated projective R-module does not support, in general, a reachable system, see [10, p. 174]. However, it is clear that a basic submodule is always the support of a non-trivial linear system. For this reason, we are interested in finding out how many poles can be assigned to a non-necessarily reachable linear system.

We refer to [1,9] for a general reading on the Theory of Linear Systems over commutative rings. However, sometimes we need to work with linear systems over projective modules; the reader can see [5] for definition and properties.

The starting point is the Classical Pole-Shifting Theorem over a field k, see [9, Theorem 7]. A linear system Σ is a pair (A, B), where A is an $n \times n$ matrix and B is an $n \times m$ matrix, both with coefficients in k.

Theorem 1.1 (Classical pole-shifting). *Let $\Sigma = (A, B)$ be a linear system over k and let r be the rank of the matrix $(B|AB|\cdots|A^{n-1}B)$. For $\lambda_1, \ldots, \lambda_r$ of k there exists a matrix F such that*

$$\chi\,(A + BF) = (x - \lambda_1)\cdots(x - \lambda_r)\cdot g(x),$$

where $\chi(A + BF)$ is the characteristic polynomial of $A + BF$ and $g(x)$ is an invariant of Σ. In this case, we say that r poles can be assigned to Σ.

Now suppose that $\Sigma = (A, B)$ is a linear system over a commutative ring R. For every maximal ideal $\mathfrak{m}$ of R let $\Sigma(\mathfrak{m}) = (A(\mathfrak{m}), B(\mathfrak{m}))$ be the extension of Σ to the residual field $R/\mathfrak{m}$. If $r(\Sigma)$ denotes the number of poles that we can assign to Σ, then it is clear that $r(\Sigma) \leqslant r(\Sigma(\mathfrak{m}))$. By the Classical Pole-Shifting Theorem, we have that

$$r\,(\Sigma(\mathfrak{m})) \geqslant \operatorname{rank}\Big(B\,(\mathfrak{m})\,|A(\mathfrak{m})B(\mathfrak{m})|\cdots|A(\mathfrak{m})^{n-1}B(\mathfrak{m})\Big).$$

In Section 2, we introduce the feedback invariant associated to Σ given by

$$\operatorname{res.rk}(\Sigma) = \min\Big\{\operatorname{rank}\Big(B(\mathfrak{m})|A(\mathfrak{m})B(\mathfrak{m})|\cdots|A(\mathfrak{m})^{n-1}B(\mathfrak{m})\Big) : \mathfrak{m} \in \operatorname{Max}(R)\Big\}.$$

We say that R is a PS ring if $\operatorname{res.rk}(\Sigma) \leqslant r(\Sigma)$ for every linear system Σ. Since Σ is reachable if and only if $\operatorname{res.rk}(\Sigma) = n$, then it follows that a PS ring is a PA ring. The main result of this section is that a bcs ring is a PS ring.

An old problem is, see [2]: Does every Prüfer domain have the bcs property? We prove that over a Prüfer domain the properties bcs ring and PS ring are equivalent. We also prove that if R is a Bezout domain, then R is an elementary divisor domain if and only if R is a PS ring. Note that an old conjecture related to the pole assignability, see [1, p. 92], is that every Bezout domain is an elementary divisor domain.

Section 3 is devoted to study the Converse Pole-Shifting, that is to say, when $r(\Sigma) \leqslant \operatorname{res.rk}(\Sigma)$. A classical result in Control Theory shows that if Σ is pole assignable ($r(\Sigma) = n$), then $\operatorname{res.rk}(\Sigma) = n$. We give examples when $\operatorname{res.rk}(\Sigma) < r(\Sigma)$. The main result is: over a noetherian ring, the Converse Pole-Shifting holds if and only if R is residually infinite.

2. Residual rank and pole-shifting

Let R be a commutative ring with identity element and let M be a finitely generated projective R-module with constant finite rank n. A linear system over M is a pair $(A, \mathscr{B})$, where $A : M \to M$ is an endomorphism and $\mathscr{B}$ is a finitely generated submodule of M. The system $(A, \mathscr{B})$ is reachable if the submodule

$$\mathscr{B} + A\mathscr{B} + \cdots + A^i\mathscr{B} + \cdots$$

is M. Note that by the Cayley–Hamilton Theorem the above submodule is equal to $\mathscr{B} + A\mathscr{B} + \cdots + A^{n-1}\mathscr{B}$.

In the sequel, we always specify $\mathscr{B}$ in terms of generators giving a linear map $B : R^m \to M$ such that the image of B is $\mathscr{B}$. Hence a linear system over M will be given by a pair $\Sigma = (A, B)$, where $A : M \to M$ is an endomorphism and $B : R^m \to M$ is a homomorphism. Now a reachable system is one for which the following homomorphism

$$\begin{aligned} A * B : R^m \oplus \overset{n)}{\cdots} \oplus R^m &\to M \\ (u_1, \ldots, u_n) &\to \sum_{i=1}^{n} A^{i-1} B u_i, \end{aligned}$$

is onto. If $M = R^n$ is free and we fix the standard bases of R^m and R^n, the linear maps A and B will be given by matrices that we also denote by A and B; in this case the homomorphism $A * B$ is given by the block matrix

$$\left(B|AB|\cdots|A^{n-1}B\right).$$

We will use without distinction the notation $A * B$ for the homomorphism and for the matrix.

For a maximal ideal $\mathfrak{m}$ of R let $\pi_{\mathfrak{m}}$ be the canonical homomorphism from R to the residual field $R/\mathfrak{m}$. We denote by $\Sigma(\mathfrak{m})$ the n-dimensional system over $R/\mathfrak{m}$ given by change of scalars from R to $R/\mathfrak{m}$ via $\pi_{\mathfrak{m}}$ (i.e., $\Sigma(\mathfrak{m}) = (A(\mathfrak{m}), B(\mathfrak{m}))$, where $A(\mathfrak{m}) = A \otimes_R \mathrm{Id}_{R/\mathfrak{m}}$ and $B(\mathfrak{m}) = B \otimes_R \mathrm{Id}_{R/\mathfrak{m}}$), see [4].

Definition 2.1. The residual rank of Σ is defined by

$$\text{res. rk}(\Sigma) = \min\left\{\text{rank}\,(A(\mathfrak{m}) * B(\mathfrak{m})) : \mathfrak{m} \in \text{Max}(R)\right\}.$$

If Σ is a free system ($M = R^n$), then

$$\text{res. rk}(\Sigma) = \max\left\{\nu \in \mathbb{Z}^+ : \mathscr{U}_\nu(A * B) = R\right\},$$

where $\mathscr{U}_\nu(A * B)$ denotes the ideal generated by all the $\nu \times \nu$—minors of the matrix $A * B$.

Proposition 2.2. *Suppose that* $\Sigma = (A, B)$ *is feedback equivalent to* $\Sigma' = (A', B')$. *Then* $\text{res. rk}(\Sigma) = \text{res. rk}(\Sigma')$.

Proof. It is sufficient to prove that for every maximal ideal $\mathfrak{m}$ of R we have the equality $\operatorname{rank}(A(\mathfrak{m}) * B(\mathfrak{m})) = \operatorname{rank}(A'(\mathfrak{m}) * B'(\mathfrak{m}))$. This follows from [6, Lemma 2.1]. $\square$

Recall that a submodule $\mathscr{B}$ of a finitely generated projective R-module M is basic if locally $\mathscr{B}$ contains a non-trivial summand of M, or equivalently, if the image of $\mathscr{B}$ in $M/\mathfrak{m}M$ is non-zero for each maximal ideal $\mathfrak{m}$ of R. If $M = R^n$ is free and B is an $n \times m$ matrix whose columns span $\mathscr{B}$, then the submodule $\mathscr{B}$ is basic if and only if $\mathscr{U}_1(B) = R$.

Lemma 2.3. *Let $\Sigma = (A, B)$ be a linear system over M. Then* $\operatorname{res.rk}(\Sigma) \geqslant 1$ *if and only if the image of B is a basic submodule of M.*

Proof. $\operatorname{res.rk}(\Sigma) = 0$ if and only if there exists a maximal ideal $\mathfrak{m}$ of R such that $A(\mathfrak{m}) * B(\mathfrak{m}) = 0$, or equivalently, $B(\mathfrak{m}) = 0$. $\square$

Definition 2.4. A commutative ring R has the BCS property (or R is a bcs ring) if every basic submodule of a finitely generated projective R-module M contains a rank 1 summand of M.

Definition 2.5. Let $\Sigma = (A, B)$ be a linear system over M with $B : R^m \to M$. We say that r poles can be assigned to Σ if for $\lambda_1, \ldots, \lambda_r$ of R there exists $F : M \to R^m$ such that

$$\chi(A + BF) = (x - \lambda_1) \cdots (x - \lambda_r) \cdot g(x),$$

where $\chi(A + BF)$ denotes the characteristic polynomial of $A + BF$.

Definition 2.6. The ring R is pole-shifting (or R is a PS ring) if $\operatorname{res.rk}(\Sigma)$ poles can be assigned to Σ for every system Σ. A PSF ring is one for which $\operatorname{res.rk}(\Sigma)$ poles can be assigned to Σ for every free system Σ.

Since $\operatorname{res.rk}(\Sigma) = n$ if and only if Σ is reachable it follows that a PS ring is a PA ring.

Theorem 2.7. *Let R be a bcs ring. Then R is a PS ring.*

Proof. Let R be a bcs ring and let $\Sigma = (A, B)$ be a linear system over a finitely generated projective R-module M of rank n. Suppose that $\operatorname{res.rk}(\Sigma) \geqslant 1$. Then, by Lemma 2.3, $\operatorname{Im}(B)$ is basic and hence there exists a rank one direct summand P_1 of M such that $P_1 \subseteq \operatorname{Im}(B)$. Put $M = P_1 \oplus P_2$ and let π_i be the canonical projection of M onto P_i for $i = 1, 2$. Since $P_1 \subseteq \operatorname{Im}(B)$ it follows that $\pi_1 B : R^m \to P_1$ is onto and consequently there exists a rank one direct summand P_1' of R^m isomorphic via $\pi_1 B$ to P_1. Put $R^m = P_1' \oplus P_2'$, where $P_2' = \ker(\pi_1 B)$. With respect to the decompositions $R^m = P_1' \oplus P_2'$ and $M = P_1 \oplus P_2$ the system Σ is defined by

$$\Sigma = \left(A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right),$$

where $b_{ij} = (\pi_i B)|_{P_j'}$ is the restriction to P_j' of $\pi_i B$ and $a_{ij} = (\pi_i A)|_{P_j}$ is the restriction to P_j of $\pi_i A$.

By construction b_{11} is an isomorphism and b_{12} is zero. Using a suitable feedback action we have that Σ is feedback equivalent to the system

$$\Sigma' = \left(A' = \begin{pmatrix} 0 & 0 \\ a_{21}' & a_{22}' \end{pmatrix},\ B = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix} \right).$$

We now prove the result by induction on $\operatorname{res.rk}(\Sigma)$. Suppose that $\operatorname{res.rk}(\Sigma) = 1$ and let f_λ be the endomorphism of P_1 such that $\chi(f_\lambda) = (x - \lambda)$. Since Σ' is feedback equivalent to

$$\Sigma'' = \left(A'' = \begin{pmatrix} f_\lambda & 0 \\ a_{21}' & a_{22}' \end{pmatrix},\ B = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix} \right),$$

it follows that $\chi(A'') = (x - \lambda) \cdot \chi(a_{22}')$ and hence one pole can be assigned to Σ.

Suppose that $\operatorname{res.rk}(\Sigma) > 1$. Let Γ be the linear system over P_2 given by $\Gamma = (a_{22},\ (b_{22}|a_{21}'b_{11}))$. Since

$$A' * B = P_1 \oplus (a_{22}' * (b_{22}|a_{21}'b_{11})),$$

then $\operatorname{res.rk}(\Gamma) = \operatorname{res.rk}(\Sigma) - 1$ and therefore the result follows by induction. $\square$

Corollary 2.8. *The following classes of rings are bcs rings and hence PS rings*:

(i) *Elementary divisor rings.*
(ii) *Semilocal rings.*
(iii) *Dedekind domains.*
(iv) *Both the rings* $\mathscr{C}^0(X;\mathbb{R})$ *of continuous and* $\mathscr{C}^\infty(X;\mathbb{R})$ *of differentiable real valued functions over the connected manifold* X, *where* $\dim(X) \leqslant 1$.
(v) 0-*Dimensional rings.*
(vi) 1-*Dimensional domains.*
(vii) *The polynomial ring* $V[x]$, *where* V *is a semilocal principal ideal domain.*
(viii) *The ring* $\mathscr{C}^0(X;\mathbb{C})$ *of continuous complex valued functions over the connected manifold* X, *where* $\dim(X) \leqslant 2$.
(ix) 1-*Dimensional rings.*

Proof. See [1] for (i). See [1] or [10] for (ii). See [5] for (iii) and (iv). See [10] for (v)–(vii). See [11] for (viii). Finally see [3] for (ix). $\square$

It is known that if R is a bcs ring, then every quotient R/I is a bcs ring, see [10]. In the same way it is known that if R is a PAF ring (reachable free systems are pole assignable), then every quotient R/I is a PAF ring, see [2]. The referee has suggested the following result:

Theorem 2.9. *Let I be an ideal of R. If R is a PSF ring, then R/I is a PSF ring.*

Proof. Let $\pi : R \to R/I$ be the canonical homomorphism, let $\overline{\Sigma} = (\overline{A}, \overline{B})$ be a free n-dimensional system over R/I with res. rk$(\Sigma) = r$, and let $\pi(\lambda_1), \ldots, \pi(\lambda_r)$ be elements of R/I. Choose a free n-dimensional system $\Sigma = (A, B)$ such that $\pi(A) = \overline{A}$ and $\pi(B) = \overline{B}$. Clearly

$$\mathscr{U}_r(\overline{A} * \overline{B}) = \mathscr{U}_r(A * B) \cdot R/I.$$

Thus there exist $r \times r$ minors $b_1, \ldots, b_t$ of $A * B$ and elements $\alpha_1, \ldots, \alpha_t$ of R such that

$$\pi\ (\alpha_1 b_1 + \cdots + \alpha_t b_t) = \overline{1}.$$

Put $a = 1 + \sum \alpha_i b_i \in I$ and consider the $n \times (m + r)$ matrix

$$B_1 = \left(B|a\underline{e_1}|a\underline{e_2}| \cdots |a\underline{e_r}\right),$$

where $\underline{e_i} = (0, \ldots, 1, \ldots, 0)^{\mathrm{t}}$ is the ith standard basis vector of R^n. Since the ideals $\mathscr{U}_r(A * B)$ and (a^r) are contained in $\mathscr{U}_r(A * B_1)$ it follows that $\mathscr{U}_r(A * B_1) = R$.

Now, as R is a PS ring, there exists a matrix $K = \binom{K_1}{K_2}$ such that

$$\chi(A + B_1 K) = (x - \lambda_1) \cdots (x - \lambda_r) g(x).$$

Taking images under the map π, we get

$$\begin{aligned}\chi\left(\overline{A} + (\overline{B}|0)\begin{pmatrix}\pi\ (K_1)\\ \pi\ (K_2)\end{pmatrix}\right) &= \chi\left(\overline{A} + \overline{B} \cdot \pi\ (K_1)\right)\\ &= (x - \pi\ (\lambda_1)) \cdots (x - \pi\ (\lambda_r)) \cdot \pi\ (g(x)). \qquad \square\end{aligned}$$

A domain R is Prüfer if every non-zero finitely generated ideal I of R is invertible (i.e., there exists an R-submodule I^{-1} of the quotient field $K(R)$ of R such that $I \cdot I^{-1} = R$). An old problem is: Does every Prüfer domain have the BCS property? In [2] it is proved that a Prüfer domain is a bcs ring if and only if R has the Simultaneous Basis Property. We next characterize the Prüfer domains that are bcs rings in terms of the Pole-Shifting Property.

Theorem 2.10. *Let R be a Prüfer domain. Then the following conditions are equivalent*:

(i) *R is a bcs ring.*
(ii) *R is a PS ring.*

Proof. It is sufficient to prove that if B is an $n \times m$ matrix with $\mathscr{U}_1(B) = R$, then there exists a matrix K such that $\mathscr{U}_1(BK) = R$ and $\mathscr{U}_2(BK) = (0)$ (i.e., BK is an $(*)$-matrix).

Consider the n-dimensional free system $\Sigma = (0, B)$ over R , where 0 is the $n \times n$ zero matrix and $\mathscr{U}_1(B) = R$. It follows that res. rk$(\Sigma) \geqslant 1$. Therefore there exists an $m \times n$ matrix K' such that

$$\chi\left(BK'\right) = (x-1)\, g(x).$$

By McCoy's Theorem [8, Chapter 3, Theorem 6], there exists a non-zero element $\underline{v} = (v_1, \ldots, v_n)^{\mathrm{t}} \in R^n$ such that $BK'\underline{v} = \underline{v}$. Since R is a Prüfer domain, we have $\mathscr{U}_1(\underline{v}) \cdot \mathscr{U}_1(\underline{v})^{-1} = R$. Therefore there exist elements $\lambda_1, \ldots, \lambda_n$ of $\mathscr{U}_1(\underline{v})^{-1}$ such that

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = 1,$$

and $\lambda_i v_j \in R$ for every i, j. Consequently the following matrix over R

$$BK' \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \begin{pmatrix} \lambda_1 & \ldots & \lambda_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \begin{pmatrix} \lambda_1 & \ldots & \lambda_n \end{pmatrix},$$

is an $(*)$-matrix. □

A Bezout domain is a Prüfer domain in which all finitely generated projective modules are free. We next consider this case but before that we need the following result:

Lemma 2.11. *For a ring R the following statements are equivalent*:

(i) *R is a bcs ring and projective R-modules of finite rank are free.*

(ii) *For every matrix B with $\mathscr{U}_1(B) = R$ there exist invertible matrices P and Q such that*

$$PBQ = \begin{pmatrix} 1 & \underline{0}^{\mathrm{t}} \\ \underline{0} & B_1 \end{pmatrix}.$$

(iii) *Each linear system Σ with* res. rk$(\Sigma) \geqslant 1$ *is feedback equivalent to one of the form*

$$\widetilde{\Sigma} = \left(\widetilde{A} = \left(\begin{array}{cccc} 0 & \cdots & \cdots & 0 \\ \hline & & & \\ & & * & \\ & & & \end{array} \right),\ \widetilde{B} = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array} \right) \right).$$

Proof. The equivalence between (ii) and (iii) is straightforward. We prove that (i) and (ii) are equivalent. Suppose that (i) holds. Let B be an $n \times m$ matrix with $\mathscr{U}_1(B) = R$ and consider the n-dimensional free system $\Sigma = (0, B)$ over R. Following the proof of Theorem 2.7 the system Σ is feedback equivalent to $\Sigma' = (A', B')$, where B' is defined by the matrix of homomorphisms

$$\begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}.$$

The result follows taking bases of the free R-modules P_1, P_2, P_1' and P_2'.

Conversely suppose that (ii) holds. It is clear that R is a bcs ring. Let M be a projective finitely generated R-module of finite rank r and let

$$R^m \overset{\varphi}{\to} R^n \to M \to 0$$

be a presentation of M. Let B be the matrix of φ with respect to the standard bases. By [8, Chapter 4, Theorem 18] we have

$$\mathscr{U}_1(B) = \mathscr{U}_2(B) = \cdots = \mathscr{U}_{n-r}(B) = R,$$

and

$$\mathscr{U}_{n-r+1}(B) = \mathscr{U}_{n-r+2}(B) = \cdots = (0).$$

By (ii) there exist invertible matrices P and Q such that

$$PBQ = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}.$$

Therefore M is free. $\square$

Theorem 2.12. *Let R be a Bezout domain. Then the following statements are equivalent*:

(i) *R is a PS ring.*
(ii) *R is a bcs ring.*
(iii) *R is an elementary divisor domain.*

Proof. It is sufficient to prove that a Bezout domain having the PS property is an elementary divisor domain. Let B be an $n \times m$ matrix over R. Put $B = d \cdot B'$ where d is a generator of $\mathscr{U}_1(B)$ and $\mathscr{U}_1(B') = R$. By Lemma 2.11 there exist invertible matrices P and Q such that

$$PBQ = d \cdot PB'Q = d \cdot \begin{pmatrix} 1 & \underline{0}^{\mathrm{t}} \\ \underline{0} & B_1 \end{pmatrix} = \begin{pmatrix} d & \underline{0}^{\mathrm{t}} \\ \underline{0} & d \cdot B_1 \end{pmatrix},$$

and the result follows by induction. $\square$

Remark 2.13. Whether or not every Bezout domain is pole assignable is an open problem related to the old conjecture: R is a Bezout domain if and only if R is an elementary divisor domain, see [7] and [1, p. 92]. If this conjecture is true, then every Bezout domain is a bcs ring and hence a PS ring and a PA ring.

3. Converse pole-shifting results

Let $\Sigma = (A, B)$ be a free n-dimensional linear system over R. A classical result in Control Theory is that if n poles can be assigned to Σ (that is, Σ is pole assignable), then $\operatorname{res.rk}(\Sigma) = n$ (that is, Σ is reachable). However, we have the following:

Examples 3.1.

(i) Let $\mathbb{F}_q$ be the finite field with q elements. Consider the q-dimensional system over $\mathbb{F}_q$ given by $\Sigma = (D, \underline{0})$, where D is the diagonal matrix that has all the elements of $\mathbb{F}_q$ on the main diagonal. Then $\operatorname{res.rk}(\Sigma) = 0$ and however 1 pole can be assigned to Σ.

(ii) Consider the 2-dimensional free system over $\mathbb{Z}$ given by

$$\Sigma' = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right).$$

Newly $\operatorname{res.rk}(\Sigma) = 0$ and however 1 pole can be assigned to Σ.

We say that R is a residually infinite ring if and only if for every maximal ideal $\mathfrak{m}$ of R the residual field $R/\mathfrak{m}$ is infinite. Note that if R contains an infinite field, then R is a residually infinite ring.

Lemma 3.2. *Let R be a residually infinite ring and let $\Sigma = (A, B)$ be a linear system. Assume that r poles can be assigned to Σ. Then* $\operatorname{res.rk}(\Sigma) \geqslant r$.

Proof. Suppose that r poles can be assigned to $\Sigma = (A, B)$. If $\mathfrak{m}$ is a maximal ideal of R, then r poles can be assigned to $\Sigma(\mathfrak{m}) = (A(\mathfrak{m}), B(\mathfrak{m}))$.

Assume that $\operatorname{rk}(A(\mathfrak{m}) * B(\mathfrak{m})) = s < r$. Then, by Theorem 1.1, for $\lambda_1, \ldots, \lambda_s$ of $R/\mathfrak{m}$ there exists a matrix F such that

$$\chi\,(A(\mathfrak{m}) + B(\mathfrak{m})F) = (x - \lambda_1) \cdots (x - \lambda_s) \cdot g(x),$$

where $g(x)$ is independent of $\lambda_1, \ldots, \lambda_s$. Since $R/\mathfrak{m}$ is infinite and $s < r$, then we can choose elements $\mu_1, \ldots, \mu_r$ of $R/\mathfrak{m}$ such that μ_i is not a root of $g(x)$. It is clear that $\mu_1, \ldots, \mu_r$ cannot be assigned to $\Sigma(\mathfrak{m})$. □

Theorem 3.3. *Let R be a noetherian ring. Then the following statements are equivalent*:

(i) *If Σ is a free system such that r poles can be assigned to Σ, then* $\operatorname{res.rk}(\Sigma) \geqslant r$.

(ii) *R is a residually infinite ring.*

Proof. By Lemma 3.2 it is sufficient to prove that (i) implies (ii). Suppose that there exists a maximal ideal $\mathfrak{m}$ of R such that $R/\mathfrak{m}$ is finite. There exist elements $\mu_1, \ldots, \mu_t$ of R such that $R/\mathfrak{m} = \{\pi_{\mathfrak{m}}(\mu_1), \ldots, \pi_{\mathfrak{m}}(\mu_t)\}$, where $\pi_{\mathfrak{m}}(\mu_i)$ is the canonical image of μ_i in $R/\mathfrak{m}$.

Since R is noetherian, then $\mathfrak{m}$ is finitely generated. Let $\{g_1, \ldots, g_r\}$ be a set of generators of $\mathfrak{m}$ and let $\Sigma = (A, B)$ be the t-dimensional free linear system over R given by

$$\Sigma = \left(\begin{pmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mu_t \end{pmatrix}, \begin{pmatrix} g_1 & g_2 & \cdots & g_r \\ g_1 & g_2 & \cdots & g_r \\ \vdots & \vdots & \ddots & \vdots \\ g_1 & g_2 & \cdots & g_r \end{pmatrix} \right).$$

We will prove that $\operatorname{res.rk}(\Sigma) = 0$ and however 1 pole can be assigned to Σ.

Clearly $\operatorname{res.rk}(\Sigma) = 0$ because all elements of B are in $\mathfrak{m}$. Let λ be an element of R. Then $\pi_{\mathfrak{m}}(\lambda) = \pi_{\mathfrak{m}}(\mu_i)$ for some i and hence

$$\lambda - \mu_i = \alpha_1 g_1 + \cdots + \alpha_r g_r,$$

where $\alpha_1, \ldots, \alpha_r$ are elements of R. Put

$$F = \begin{pmatrix} 0 & \cdots & 0 & \alpha_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \alpha_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \alpha_r & 0 & \cdots & 0 \end{pmatrix},$$

where all entries are zero except for the ith column. Then, λ is a root of $\chi(A + BF)$ and consequently 1 pole can be assigned to Σ. $\square$

The above theorem does not hold for an arbitrary non-noetherian ring. We next give an example of this fact.

Example 3.4. Let R be a ring such that (a) the ring R is not noetherian; (b) the ring R is not residually infinite; (c) for each finitely generated ideal $\mathfrak{a}$ of R, there exists a maximal ideal $\mathfrak{m}$ of R containing $\mathfrak{a}$ such that the residual field $R/\mathfrak{m}$ is infinite.

We claim that $\operatorname{res.rk}(\Sigma) \geqslant r$ for every free linear system $\Sigma = (A, B)$ over R such that r poles can be assigned to Σ.

Suppose that $\operatorname{res.rk}(\Sigma) < r$. Put $\mathfrak{a} = \mathscr{U}_r(A * B)$. Let $\mathfrak{m}$ be a maximal ideal of R containing the ideal $\mathfrak{a}$ and such that $R/\mathfrak{m}$ is infinite. Then the system $\Sigma(\mathfrak{m}) = (A(\mathfrak{m}), B(\mathfrak{m}))$ verifies that $\mathscr{U}_r(A(\mathfrak{m}) * B(\mathfrak{m})) = (0)$, which is a contradiction because r poles can be assigned to $\Sigma(\mathfrak{m})$.

Note that an example of a ring verifying (a)–(c) is the polynomial ring $\mathbb{F}_q[x_1, x_2, \ldots, x_n, \ldots]$ in a countably many indeterminates over the field with q elements.

Acknowledgement

We are very grateful to the referee for his profitable comments. The proof of Theorem 2.9 is due to the referee.

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